Sigma models and generalized geometries

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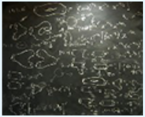

Based on:

1311.4878 (NPB) and 1505.05457 (JHEP) with **L. Jonke** and **O. Lechtenfeld**

in progress with **O. Lechtenfeld** and **R.J. Szabo**

Motivation

- T-duality in string theory
  - Symmetry between two 2D sigma models with different target space
  - Captured by an "intermediate" gauge theory ("Buscher procedure")
  - or by two different 1st order actions ("Duff procedure")
  - What is the most systematic description of dual string actions?

- "Non-geometric" string backgrounds; e.g. $H_{ijk} \to f^{i}_{jk} \to Q^{ij}_{k} \to R^{ijk}$
  - Attractive features for flux compactifications
  - Require generalized geometric concepts
  - Usually studied from a target space viewpoint
  - World-volume description of non-geometric fluxes?

An underlying mathematical theme

- T-duality is an isomorphism between certain structures called Courant algebroids
  Bouwknegt, Hannabuss, Mathai '03; Cavalcanti, Gualtieri '11
- Given such a structure one can uniquely reconstruct a membrane sigma model
  Hofman, J. .S. Park '02; Roytenberg '06
- Non-Geometry $\leftrightsquigarrow$ Generalized Geometry $\leftrightsquigarrow$ Courant Algebroids
  ...

3D sigma models $\Leftrightarrow$ non-geometric string backgrounds

The good old Duff approach to duality rotations

Duff '89; Duff, Lu '90

Suppose we have 2D bosonic strings, in a constant background metric and B-field

$$S_0 = \int_{\Sigma_2} \left(\tfrac{1}{2} g_{ij} \mathrm{d}X^i \wedge * \mathrm{d}X^j + \tfrac{1}{2} B_{ij} \mathrm{d}X^i \wedge \mathrm{d}X^j \right) \ .$$

Shift to a 1st order formulation, with a parameter (1-form) $q \in \Omega^1(\Sigma_2, X^* TM)$:

$$S_X = \int_{\Sigma_2} \left(-\tfrac{1}{2} g_{ij} q^i \wedge * q^j - \tfrac{1}{2} B_{ij} q^i \wedge q^j + g_{ij} q^i \wedge * \mathrm{d}X^j + B_{ij} q^i \wedge \mathrm{d}X^j \right) \ .$$

A 2nd 1st order formulation, again with a parameter q and this time with variables $\widetilde{X}_i$:

$$S_{\widetilde{X}} = \int \left(\tfrac{1}{2} g_{ij} q^i \wedge * q^j + \tfrac{1}{2} B_{ij} q^i \wedge q^j + \mathrm{d}\widetilde{X}_i \wedge q^i \right) \ .$$

Duality rotation and a nice formula

Duality of the two formulations:

$$\begin{aligned} \mathrm{d}\widetilde{X}_i &= g_{ij} * \mathrm{d}X^j + B_{ij}\mathrm{d}X^j \,, \\ \mathrm{d}X^i &= P^{ij} * \mathrm{d}\widetilde{X}_j + Q^{ij}\mathrm{d}\widetilde{X}_j \,, \end{aligned}$$

with

$$P = -(g - Bg^{-1}B) \,, \qquad Q = (Bg^{-1})P \,.$$

Now if you collect $\mathbb{X}^I = (X^i, \widetilde{X}_i)$, you reach the following nice formula

$$\eta_{IJ}\mathrm{d}\mathbb{X}^J = \mathcal{H}_{IJ} * \mathrm{d}\mathbb{X}^J \,,$$

where the two metrics are

$$\eta_{IJ} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \,, \qquad \mathcal{H}_{IJ} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \,.$$

The relevance of these metrics was rediscovered $>$ 15y later in generalized geometry.

Why go up to 3D then?

- Fluxes (or Wess-Zumino terms) anyway live one dimension higher
  - + the Duff approach technically does not consider the presence of fluxes
- $\mathrm{d}X^i \to q^i$ and $\mathrm{d}\widetilde{X}_i \to p_i$, and η_{IJ} are systematically described in 3D sigma models

The membrane bosonic action

$$S_{\text{membrane}} = \int_{\Sigma_3} \left(F_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\eta_{IJ} A^I \wedge \mathrm{d}A^J - \rho^i_I(X) A^I \wedge F_i + \tfrac{1}{6} T_{IJK} A^I \wedge A^J \wedge A^K \right) .$$

$X^i : \Sigma_3 \to$ M worldvolume scalars F_i: auxiliary worldvolume 2-form ρ^i_I: components of a map

$A^I = (q^i, p_i)$: generalized 1-form η_{IJ}: the O(d,d)-invariant metric. T: generalized 3-form.

The 3-form T_{IJK} systematically includes all types of 3-elements $H_{ijk}, f^i_{jk}, Q^{ij}_k, R^{ijk}$.

At the same time, the 1-form A^I systematically includes the first order variables q^i, p_i.

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In an index-free spirit

$$S_{\text{membrane}} = \int_{\Sigma_3} \Big(\langle F, \mathrm{d}X \rangle + \tfrac{1}{2}\langle A, \mathrm{d}A\rangle_E - \langle F, \rho(A)\rangle + \langle A, [A, A]_E\rangle_E \Big) ,$$

E some vector bundle (e.g. $TM \oplus T^*M$), $\quad \langle\cdot,\cdot\rangle_E$ non-degenerate symmetric bilinear form,
$[\cdot,\cdot]_E$ a bracket of sections, $\quad \rho : E \to TM$ a homomorphism $\quad \langle\cdot,\cdot\rangle$ the standard inner product

Such a structure (obeying certain axioms) is called a Courant algebroid.

The boundary of the membrane

For the purposes of string theory, the membrane has a boundary, $\Sigma_2 = \partial\Sigma_3$.

The Courant sigma model is topological; add dynamics in the boundary action, e.g.

$$S_{\partial\Sigma_3,\text{kinetic}} = \int_{\partial\Sigma_3} \tfrac{1}{2} g_{ij} \mathrm{d}X^i \wedge *\mathrm{d}X^j \quad \text{or even} \quad \int_{\partial\Sigma_3} \tfrac{1}{2} \mathcal{H}_{IJ} A^I \wedge *A^J \ .$$

It is often convenient to also add a general topological boundary term

$$S_{\partial\Sigma_3,\text{topological}} = \int_{\partial\Sigma_3} \tfrac{1}{2} \mathcal{B}_{IJ}(X) A^I \wedge A^J \ .$$

Consistency imposes bulk/boundary conditions, which in turn yield

- general expressions for the fluxes,
- integrability conditions for “Dirac structures” $\rightsquigarrow$ physical target space.

The 2D theories belong to the large class of so-called Dirac sigma models.

Kotov, Schaller, Strobl ’04

A familiar case: NSNS H flux on a torus

$$S_H = \int_{\Sigma_3} \left(F_i \wedge \mathrm{d}X^i + q^i \wedge \mathrm{d}p_i - q^i \wedge F_i + \tfrac{1}{6} H_{ijk} q^i \wedge q^j \wedge q^k \right) + \int_{\Sigma_2} \tfrac{1}{2} g^{ij} p_i \wedge * p_j \ .$$

Field equation for F_i: $\quad q^i = \mathrm{d}X^i$.

$$\rightsquigarrow \quad S_H = \int_{\Sigma_2} \left(p_i \wedge \mathrm{d}X^i + \tfrac{1}{2} g^{ij} p_i \wedge * p_j \right) + \int_{\Sigma_3} \tfrac{1}{6} H_{ijk} \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ .$$

Field equation for p_i: $\quad p_i = -g_{ij} * \mathrm{d}X^j$.

$$\rightsquigarrow \quad S_H = \int_{\Sigma_2} \tfrac{1}{2} g_{ij} \mathrm{d}X^i \wedge * \mathrm{d}X^j + \int_{\Sigma_3} \tfrac{1}{6} H_{ijk} \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ . \quad \checkmark$$

Membrane model for R flux?

Mylonas, Schupp, Szabo '12

Let the membrane action be

$$S_R = \int_{\Sigma_3} \Big(F_i \wedge \mathrm{d}X^i + q^i \wedge \mathrm{d}p_i - q^i \wedge F_i + \tfrac{1}{6} R^{ijk} p_i \wedge p_j \wedge p_k \Big) + \int_{\Sigma_2} \tfrac{1}{2} g^{ij} p_i \wedge * p_j \ .$$

Field equation for F_i: $\quad q^i = \mathrm{d}X^i$.

$$\rightsquigarrow \quad S_R = \int_{\Sigma_2} \big(p_i \wedge \mathrm{d}X^i + \tfrac{1}{2} g^{ij} p_i \wedge * p_j \big) + \int_{\Sigma_3} \tfrac{1}{6} R^{ijk} p_i \wedge p_j \wedge p_k \ .$$

Field equation for X^i (assuming R^{ijk}, g^{ij} constant):

$$\mathrm{d}p_i = 0 \quad \Rightarrow \quad p_i = \mathrm{d}\widetilde{X}_i \quad \text{(locally), with} \quad \widetilde{X}_i \in C^\infty(\Sigma_3, X^* T^* M).$$

Very suggestive!

BUT, unfortunately no CA yields this membrane action ...

Membrane model for R flux?

Bessho, Heller, Ikeda, Watamura '15

cf. ACh, unpublished note

Let the membrane action be (for a non-degenerate Poisson 2-vector $\pi = \frac{1}{2}\pi^{ij}\partial_i \wedge \partial_j$)

$$S_R = \int_{\Sigma_3} \Big(F_i \wedge \mathrm{d}X^i + p_i \wedge \mathrm{d}q^i - \pi^{ij} p_j \wedge F_i - \tfrac{1}{2}\partial_i \pi^{jk} q^i \wedge p_j \wedge p_k + \tfrac{1}{6} R^{ijk} p_i \wedge p_j \wedge p_k \Big) + \int_{\Sigma_2} \tfrac{1}{2} g^{ij} p_i \wedge * p_j \ .$$

Field equation for F_i: $\quad \pi^{ij} p_j = \mathrm{d}X^i$.

$$\rightsquigarrow \quad S_R = \int_{\Sigma_2} \tfrac{1}{2}\widetilde{g}_{ij} \mathrm{d}X^i \wedge * \mathrm{d}X^j + \int_{\Sigma_3} \tfrac{1}{6} R^{lmn} \pi^{-1}_{li} \pi^{-1}_{mj} \pi^{-1}_{mk} \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ .$$

This is indeed a Courant algebroid!

BUT, unfortunately it cannot be the stringy R flux background ... Halmagyi '08

One step further

ACh, Jonke, Lechtenfeld '15;

ACh, Lechtenfeld, Szabo '16

Embed the membrane theory in the $(X^i, \widetilde{X}_i)$-space from the beginning.

Motivated by the Mylonas, Schupp, Szabo attempt and by the Bouwknegt, Hannabuss, Mathai diagram

$$\begin{array}{ccccc} & & (M \times_B \widetilde{M}, p^* H - \widetilde{p}^* \widetilde{H}) & & \\ & \swarrow^{p} & & \searrow^{\widetilde{p}} & \\ (M, H) & & & & (\widetilde{M}, \widetilde{H}) \\ & \searrow_{\pi} & & \swarrow_{\widetilde{\pi}} & \\ & & B & & \end{array}$$

Membrane sigma models on correspondence spaces, e.g. $M \times_B \widetilde{M}$.

The extended bosonic action

ACh, Jonke, Lechtenfeld '15

Minimal generalization:

$$S_{\Sigma_3} = \int_{\Sigma_3} \Big(F_i \wedge \mathrm{d}X^i + \widetilde{F}^i \wedge \mathrm{d}\widetilde{X}_i + \tfrac{1}{2}\eta_{IJ} A^I \wedge \mathrm{d}A^J - \rho^i_I A^I \wedge F_i - \widetilde{\rho}_{iI} A^I \wedge \widetilde{F}^i + \tfrac{1}{6} T_{IJK} A^I \wedge A^J \wedge A^K \Big) ,$$

with the same boundary action as before, but $\mathcal{B} = \mathcal{B}(X, \widetilde{X})$.

$\widetilde{F}^i$ is also an auxiliary world volume 2-form, and there is a map $\widetilde{\rho} : E \to T^*M$.

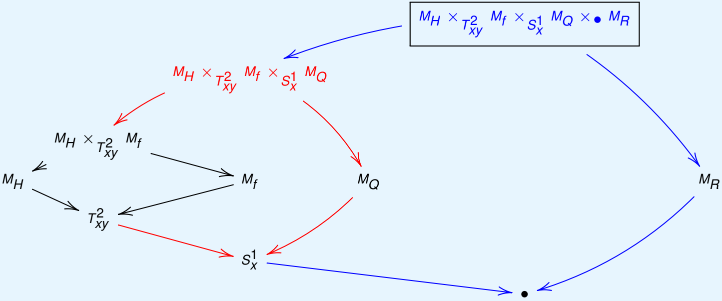
A similar analysis as before, yields extended bulk/boundary consistency conditions.

We find an apparent similarity to the flux formulation of double field theory.
Geissbuhler '11, Aldazabal, Baron, Marques, Nunez '11

Towards the full correspondence space theory

The mathematical setting of this extended action is yet to be fully understood.

The simplest geometric orbit suggests the correspondence space:



for the duality orbit:

$$H^{(0,3)} \overset{T}{\longleftrightarrow} f^{(1,2)} \overset{T}{\longleftrightarrow} Q^{(2,1)} \overset{T}{\longleftrightarrow} R^{(3,0)}$$

Epilogue

Take-home messages

- Worldvolume description of non-geometric flux backgrounds via 3D σ-models.
- A correspondence space theory is necessary to describe them all.

Some things to do, among many

- New classes of non-geometric string backgrounds?
- Relation of correspondence theory to “double field theory” or “metastring theory”?
- strings $\rightsquigarrow$ 3D σ-models $\xrightarrow{?}$ membranes $\rightsquigarrow$ 4D σ-models? U-duality?

  cf. “H-twisted Lie algebroids” Grützmann ’10 or “Lie algebroids up to homotopy” Ikeda, Uchino ’10