Sasakian Quiver Gauge Theory on the Conifold

based on joint work with O. Lechtenfeld, A. Popov, and R. Szabo
[1601.05719]

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Overview

Aims and Motivation

General Setup and Questions:

- reduction of higher-dimensional gauge theories on cosets $M^d \times G/H$: Yang-Mills theory on $M^d \times G/H \rightarrow$ Yang-Mills-Higgs theory on M^d
- coset G/H chosen with special geometry, here: Sasaki-Einstein
- systematic restrictions from imposing G-equivariance on vector bundles over $M^d \times G/H \rightarrow$ fixes form of gauge connection
- graphical encoding of the theory in quiver diagrams
- study of (generalized) instanton solutions of the gauge theory
- description of the moduli space of Hermitian Yang-Mills (HYM) instantons on metric cone $C(T^{1,1})$ by Kähler quotients and adjoint orbits

In this talk: quiver gauge theory on $G/H = T^{1,1} := \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{U}(1)}$
(JG, Lechtenfeld, Popov, Szabo [1601.05719])

Geometry of $T^{1,1}$ and the Conifold

- examples of Sasaki-Einstein manifolds in 5D: S^5 and $T^{1,1}$
- $T^{p,q} := \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{U}(1)_{\mathrm{p,q}}}$ with $\mathrm{U}(1)_{p,q} := \mathrm{span}\langle p\ I_3^{(1)} + q\ I_3^{(2)}\rangle$ (Romans 1985), $\mathrm{U}(1)$-bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1 \simeq S^2 \times S^2$
- as Sasaki-Einstein 5-manifold, $T^{1,1}$ can be described as $\mathrm{SU}(2)$ structure with (Conti, Salamon 2007)

$$\eta = -e^5, \ \omega^1 = e^{23} + e^{14}, \ \omega^2 = e^{31} + e^{24}, \text{ and } \omega^3 = e^{12} + e^{34} \quad (1)$$

satisfying

$$\mathrm{d}\eta = 2\omega^3, \quad \mathrm{d}\omega^1 = -3\eta\wedge\omega^2, \text{ and } \mathrm{d}\omega^2 = 3\eta\wedge\omega^1. \quad (2)$$

- contact form $\eta \longleftrightarrow \mathrm{U}(1)^{\perp}_{1,1}$, and ω^3 is Kähler form on $S^2 \times S^2$.
- Ricci tensor is $\mathrm{Ric}^g = 4g$
- metric cone $C\left(T^{1,1}\right)$ with metric

$$\mathrm{d}s^2_{con} := r^2\mathrm{d}s^2_{T^{1,1}} + \mathrm{d}r\otimes\mathrm{d}r \quad (3)$$

is Calabi-Yau 3-fold ("the conifold").

Equivariant vector bundles

Basic Ideas: (Alvarez-Cónsul, García-Prada 2003), for review see e.g. (Dolan, Szabo 2010)

- consider hermitian vector bundle $\pi : \mathcal{E} \longrightarrow M^d \times G/H$ of rank k ($\Rightarrow$ structure group $U(k)$) with trivial G-action on M^d
- $\mathcal{E}$ is G-equivariant iff diagram
$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\;G\,\curvearrowright\;} & \mathcal{E} \\ \pi\downarrow & & \downarrow\pi \\ M^d \times G/H & \xrightarrow{G\,\curvearrowright} & M^d \times G/H \end{array}$$
commutes.
- G-equiv. bdls. $\mathcal{E} \to M^d \times G/H \quad \longleftrightarrow \quad H$-equiv. bdls. $E \to M^d$; by induction $\mathcal{E} = G \times_H E$.
- requires representation of H on fibres $\mathcal{E}_x \simeq \mathbb{C}^k$. Assume that it stems from irred. G-representation $\mathcal{D}_{|H} = \bigoplus_j \rho_j$.
- yields *isotopical* decomposition $\mathcal{E} = \bigoplus_j \mathcal{E}_j$ and a breaking of the structure group $\mathrm{U}(k) \to \prod_j \mathrm{U}(k_j)$.
- **quiver diagram**: represent ρ_j's as vertices and maps $\in \mathrm{Hom}\left(\mathbb{C}^i, \mathbb{C}^j\right)$ given by G-action as arrow between vertices i and j.

Invariant gauge connection

- given a reductive homogeneous space G/H with generators I_μ, i.e.

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \mathfrak{m} =: \mathrm{span}\langle I_j \rangle \oplus \mathrm{span}\langle I_a \rangle \tag{4}$$

- G-invariant connection $\mathcal{A}$ on $M^d \times G/H$ can be written as
  (Bauer, Ivanova, Lechtenfeld, Lubbe 2010), (Kobayashi, Nomizu 1963)

$$\mathcal{A} \;=\; A + I_j \otimes e^j + X_a \otimes e^a \tag{5}$$

$$\text{with} \qquad [I_j, X_a] = f_{ja}^b X_b \qquad (\text{“equivariance condition”}) \tag{6}$$

- condition equivalent to bundle equivariance
- if H maximal torus of $G \Rightarrow$ quiver diagram is weight diagram of G
- construction procedure:
  1. choose irreducible representation $\mathcal{D}$ on $\mathbb{C}^k$
  2. use structure of weight diagram with vertices carrying ρ_i's
  3. connect vertices according to equivariance condition
  4. obtain block-matrix shape of $\mathcal{A}$

Examples of quiver diagrams: $\mathbb{C}P^1$

Kähler manifold $\mathbb{C}P^1 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$: **$\mathbf{A_{m+1}}$-Quiver**

e.g. (Alvarez-Cónsul, García-Prada 2001), (Popov, Szabo 2006)

choose decomposition in $(m+1)$ H-representations, i.e.

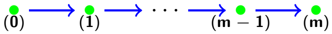
$$\mathcal{D}_{|H} = \bigoplus_{j=0}^{m} \rho_j \quad \text{on} \quad \mathbb{C}^k = \left(\mathbb{C}^{k_m}, \mathbb{C}^{k_{m-1}} \dots, \mathbb{C}^{k_0}\right)^{\mathrm{t}} \tag{7}$$

representation of H inside G is then given by the generator

$$I_3 = \mathrm{diag}\left(m\, \mathbf{1}_{k_m}, (m-2)\, \mathbf{1}_{k_{m-1}} \dots, -m\, \mathbf{1}_{k_0}\right) \tag{8}$$

equivariance condition yields quiver diagram (*"holomorphic chain"*) and gauge connection of the form

$$\mathcal{A} = \begin{pmatrix} \mathbf{1}_{k_m} \otimes a_m & \phi_{m-1} \otimes \overline{\Theta} & \mathbf{0} & \dots \\ -\phi^{\dagger}_{m-1} \otimes \Theta & \mathbf{1}_{k_{m-1}} \otimes a_{m-1} & \phi_{m-2} \otimes \overline{\Theta} & \dots \\ \mathbf{0} & -\phi^{\dagger}_{m-2} \otimes \Theta & \ddots & \dots \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

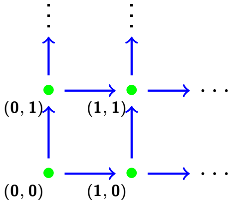


with 1-forms $a_l, \Theta \in \mathfrak{su}(2)^*$. Recall that the entries ϕ_l are matrices.

Kähler manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$: $\mathbf{A_{m_1+1} \otimes A_{m_2+1}}$**-Quiver**

(Lechtenfeld, Popov, Szabo 2008)

$\rightarrow$ yields a grading of the connection $\mathcal{A}$



Further examples e.g. on Kähler manifolds $\mathrm{SU}(3)/H$ (Lechtenfeld, Popov, Szabo 2008) and on S^5 (Lechtenfeld, Popov, Sperling, Szabo 2015)

Equivariance condition on $T^{1,1}$:

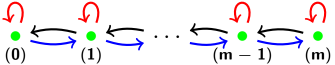
- ansatz for connection $\mathcal{A} \;=\; A + I_6 e^6 + \sum_{a=1}^{5} X_a e^a$
- yields equivariance conditions ($I_6 = I_3^{(1)} - I_3^{(2)}$)

$$[I_6, \phi^{(1)}] = 2\phi^{(1)}, \quad [I_6, \phi^{(2)}] \;=\; -2\phi^{(2)}, \quad [I_6, \psi] \;=\; 0 \tag{9}$$

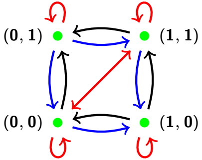
(with $\phi^{(1)} = 1/2(X_1 + \mathrm{i}\, X_2)$, $\phi^{(2)} = 1/2(X_3 + \mathrm{i}\, X_4)$ and $\psi = X_5$)

- weaker conditions than on $\mathbb{C}P^1 \times \mathbb{C}P^1 \Rightarrow$ more arrows and degeneracies

- denote by (m_1, m_2) the irreducible representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1}$
- $(m_1, m_2) = (\mathbf{m}, \mathbf{0})$ yields **"modified holomorphic chain"**



- representation $(\mathbf{1}, \mathbf{1})$



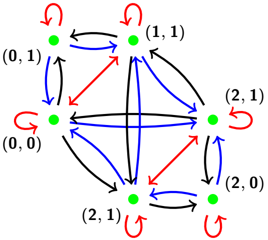
$$\mathcal{A} = \begin{pmatrix} a_{00}\mathbf{1}_{k_{00}} + \Psi_{00} & -\Phi^\dagger_{00,01} & \Phi_{10,00} & \Psi_{11,00} \\ \Phi_{00,01} & a_{01}\mathbf{1}_{k_{01}} + \Psi_{01} & 0 & \Phi_{11,01} \\ -\Phi^\dagger_{10,00} & 0 & a_{10}\mathbf{1}_{k_{10}} + \Psi_{10} & -\Phi^\dagger_{10,11} \\ -\Psi^\dagger_{11,00} & -\Phi^\dagger_{11,01} & \Phi_{10,11} & a_{11}\mathbf{1}_{k_{11}} + \Psi_{11} \end{pmatrix}$$

- form of connection also obtained by direct evaluation of equivariance condition with $I_6 = (0, 2, -2, 0)$

$$[I_6, (\bullet)] = \begin{pmatrix} 0 & -2 & 2 & 0 \\ 2 & 0 & 4 & 2 \\ -2 & -4 & 0 & -2 \\ 0 & -2 & 2 & 0 \end{pmatrix} \quad \Rightarrow \quad \phi^{(1)} = \begin{pmatrix} 0 & 0 & * & 0 \\ * & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix} \quad \psi = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix} \qquad (10)$$

Quiver diagrams on $T^{1,1}$: further examples

- representation $(\mathbf{2}, \mathbf{1})$:



- for large representations (m_1, m_2) it is advantageous to combine vertices with the same $\mathrm{U}(1)_{1,1}$ charge (i.e. identify along red arrows)
  $\Rightarrow$ modified holomorphic chain of length $(m_1 + m_2 + 1)$
- recover $\mathbb{C}P^1 \times \mathbb{C}P^1$-result by fixing $X_5 \propto I_5$ and imposing equivariance also w.r.t. second Cartan generator I_5

$$\left[I_3^{(1)} + I_3^{(2)}, \phi^{(1)}\right] = 2\phi^{(1)} \quad \text{and} \quad \left[I_3^{(1)} + I_3^{(2)}, \phi^{(2)}\right] = 2\phi^{(2)} \tag{11}$$

Generalized instantons

- generalized self-duality condition on n-dimensional manifold (Ward 1984), (Hull 1998), (Harland, Nölle 2012)

$$*\mathcal{F}_\Gamma = - * Q \wedge \mathcal{F}_\Gamma \tag{12}$$

with $(n-4)$-form Q as useful technique in Yang-Mills theory.

- leads to

$$\underbrace{\mathrm{d} * \mathcal{F} + A \wedge *\mathcal{F} - (-1)^n * \mathcal{F} \wedge A}_{\text{YM eq.}} + \underbrace{\mathrm{d} * Q \wedge \mathcal{F}}_{\text{torsion}} = 0. \tag{13}$$

- for special geometries with suitable Q this implies the Yang-Mills equation (without torsion) $\Rightarrow$ 1st-order eq. instead of 2nd-order eq.
- relation to other definitions and explicit expressions for Q given by Harland and Nölle, as well as formula for compatible connection (*"canonical connection"*)
- occurs naturally in heterotic string theory as part of the BPS equations, $\mathcal{F} \cdot \epsilon = 0$.
- on 5d Sasaki-Einstein manifold: $Q = \frac{1}{2}\omega^3 \wedge \omega^3 = e^{1234}$
on $T^{1,1}$: $\Gamma := I_6 e^6$ so $\mathcal{F}_\Gamma = 3\left(e^{12} - e^{34}\right) I_6$ and $*_5\mathcal{F}_\Gamma = -3e^5 \wedge \left(e^{12} - e^{34}\right) I_6$

Moduli Space of HYM Instantons on the conifold

- for instantons impose *Hermitian Yang-Mills equations* (also known as *Donaldson-Uhlenbeck-Yau equations*) (Donaldson 1985), (Uhlenbeck, Yau 1986), (Popov 2009)

$$\mathcal{F}^{(2,0)} = 0 = \mathcal{F}^{(0,2)} \quad \text{and} \quad \Omega \lrcorner \mathcal{F} = 0 \tag{14}$$

  on a Kähler manifold with form Ω.
- yields constraint

$$\left[\phi^{(1)}, \phi^{(2)}\right] = 0 \qquad \Rightarrow \text{commutativity of diagram} \tag{15}$$

- and (with $\tau = \ln r$, $s = 1/4\,\mathrm{e}^{-4\tau}$, $\phi^{(i)} = \mathrm{e}^{-3/2\tau} Z_i$ for $i = 1, 2$ and $\phi^{(3)} = \mathrm{e}^{-4\tau} Z_3$) the equations

$$\frac{\mathrm{d}}{\mathrm{d}s} Z_a = 2[Z_a, Z_3] \quad \text{for} \quad a = 1, 2 \quad (\textit{"complex equations"})$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(Z_3 + Z_3^\dagger\right) = 2\,(-s)^{-5/4}\left([Z_1, Z_1^\dagger] + [Z_2, Z_2^\dagger]\right) - 2[Z_3, Z_3^\dagger] \quad (\textit{"real equation"})$$

- similar to (original) Nahm equations (see e.g. Kronheimer 1984)

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = 2\,[\beta, \alpha] \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}\left(\alpha + \alpha^\dagger\right) = -2\left[\alpha, \alpha^\dagger\right] - 2\left[\beta, \beta^\dagger\right] \tag{16}$$

  $\Rightarrow$ techniques (moment maps, Kähler quotients, adjoint orbits) from the discussion of the original Nahm equations applicable (Donaldson 1984),(Kronheimer 1990)

Description of moduli space I

see (Donaldson 1984),(Kronheimer 1990), (Sperling 2015)

- denote moduli space of complex equations as $\mathbb{A}_{1,1}$
- it is invariant under the gauge transformation

$$Z_a \mapsto g Z_a g^{-1} \quad \text{for} \quad a = 1,2 \quad \text{and} \quad Z_3 \mapsto g Z_3 g^{-1} + \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}s} g^{-1}, \tag{17}$$

  for $g(s) \in \mathcal{G} \subset \mathrm{GL}(\mathbb{C}, k)$ such that constraints are satisfied.

- real equation can be considered as moment map $\mu : \mathbb{A}_{1,1} \to \mathrm{Lie}\,(\mathcal{G})$

$$\mu(Z, Z^\dagger) \;=\; \frac{\mathrm{d}}{\mathrm{d}s}\left(Z + Z^\dagger\right) - 2\,(-s)^{(-5/4)}\left(\left[Z_1, Z_1^\dagger\right] + \left[Z_2, Z_2^\dagger\right]\right) + 2\left[Z_3, Z_3^\dagger\right] \tag{18}$$

- thus moduli space $\mathcal{M}$ as Kähler quotient

$$\mathcal{M} \;=\; \mu^{-1}(0)\,/\mathcal{G} \tag{19}$$

- recall: original Nahm equations admit hyper-Kähler structure

Description of moduli space II see (Donaldson 1984),(Kronheimer 1990), (Sperling 2015)

- apply gauge transformation such that

$$Z_a = g^{-1} U_a g \text{ for } a = 1,2 \quad \text{and} \quad Z_3 = -\frac{1}{2} g^{-1} \frac{\mathrm{d}g}{\mathrm{d}s} \tag{20}$$

with constant matrices U_1 and U_2.

- real equation can be interpreted as equation of motion of a suitable Lagrangian
- one imposes the boundary conditions

$$\lim_{s\to\infty} Z_\mu(s) = g_0 T_\mu g_0^{-1} \tag{21}$$

$\Rightarrow$ determines uniquely the solutions

- thus the moduli space can be also described as adjoint orbit

$$\mathcal{M} = \mathcal{M}(T_\mu) \tag{22}$$

of constant matrices at the boundary $s \to \infty$

Summary

- G-invariant connections on G/H can be encoded in quiver diagrams $\rightarrow$ efficient tool for the construction of gauge connection
- explicit study of the 5-dimensional Sasaki-Einstein space $T^{1,1}$
- comparison with the underlying Kähler space $S^2 \times S^2$
- generalized self-duality equation as tool for Yang-Mills theory
- Hermitian Yang-Mills equations on conifold and their moduli space, techniques from discussion of Nahm's equations

Thank you for your attention!

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