Symplectic supermanifolds and non-geometric fluxes

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arXiv: 1406.3601 A.D.[1], J.Stasheff[2]
+ work in progress

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Contents

Motivation: Closed string theory

Canonical momenta and winding

- Sigma model $X : \Sigma \to M = T^d$

$$S = \int_\Sigma h^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij}\, d\mu_\Sigma + \int_\Sigma X^* B \,,$$

where $h \in \Gamma(\otimes^2 T^*\Sigma)$, $G \in \Gamma(\otimes^2 TM)$, $B \in \Gamma(\wedge^2 T^*M)$.

- Classical solutions to e.o.m. (take *closed* string $\Sigma = \mathbb{R} \times S^1$)

$$X^i_R = x^i_{0R} + \alpha^i_0(\tau - \sigma) + i \sum_{n \neq 0} \frac{1}{n} \alpha^i_n e^{-in(\tau-\sigma)} \,, \quad X^i_L = \dots \,,$$

$$\alpha^i_0 = \frac{1}{\sqrt{2}} G^{ij}\Big(p_j - (G_{jk} + B_{jk}) w^k\Big) \,,$$

  - p_k: Canonical momentum zero modes
  - w^k: *Winding* zero modes, $w^k := \frac{1}{2\pi}\int_0^{2\pi} \partial_\sigma X^k d\sigma$.

Motivation: Closed string theory

Two sets of differential operators

Siegel, Tseytlin, Hull, Zwiebach, Kugo, Hohm, Blumenhagen, Lüst, Hassler

- Two sets of momenta in α_0^i $\rightarrow$ differential operators:

$$p_i \simeq \frac{1}{i}\partial_k \ , \quad w^i \simeq \frac{1}{i}\tilde{\partial}^k \ . \tag{1}$$

- "Level matching condition" in string theory:

$$\partial_k \phi \, \tilde{\partial}^k \psi + \tilde{\partial}^k \phi \, \partial_k \psi = 0 \ , \tag{2}$$

for all elements ϕ, ψ of the algebra of observables.

Two different interpretations on observables $\phi \in C^\infty(M)$:

- $d_{dR}\phi = \partial_k \phi \, dx^k + \tilde{\partial}^k \phi \, d\tilde{x}_k$: Double configuration space, algebra of observables on it: "Double field theory".
- Here: Take Lie bialgebroid (A, A^*) and $d_A \phi = \partial_k \phi \, e^k$, $d_{A^*}\phi = \tilde{\partial}^k \phi \, e_k^*$. Make this precise and determine its relation to physics

Motivation: Generalized geometry

A word about notation   Hitchin, Gualtieri

- $O(d,d)$-transformations: $A \in \mathrm{Mat}(d,d)$,

$$A\eta A^t = \eta \ , \quad \eta_{MN} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

- Generalized vectors:

$$V = X + \xi,\ W = Y + \zeta \ \in \Gamma(TM \oplus T^*M) \ .$$

- Component notation (fundamental rep of $O(d,d)$)

$$V^M = \ (V^m(x), V_m(x)) \quad \text{and} \quad \partial^M = (\tilde{\partial}^m, \partial_m)$$

- Bilinear pairing:

$$\langle V, W\rangle = \iota_Y \xi + \iota_X \zeta \quad \text{i.e.} \quad V^M W_M = V^k W_k + V_k W^k \ .$$

Motivation: C for Courant?

The C-bracket in double field theory

Hull, Zwiebach, arXiv: 0908.1792

Double configuration space approach →: action principle + gauge symmetry. Commutator of gauge trafos: C-bracket

$$\Big([V,W]_C\Big)^M = V^K\partial_K W^M - W^K\partial_K V^M - \frac{1}{2}\Big(V^K\partial^M W_K - W^K\partial^M V_K\Big)\,. \tag{3}$$

<u>Observation for $V = X+\xi, W = Y+\zeta \in \Gamma(TM\oplus T^*M)$</u>:

$\tilde{\partial}^k = 0$: C-bracket reduces to Courant bracket.

$$[V,W]_C = [X,Y]_L + L_X\zeta - L_Y\xi + \tfrac{1}{2}d_{dR}(\iota_Y\xi - \iota_X\zeta)\,.$$

H-flux and "non-geometric" R-flux

Hull, Hohm, Zwiebach

the famous "chain"

$$H_{ijk} \to f^i_{jk} \to Q^{ij}_k \xrightarrow{?} R^{ijk}$$

1. Generalized geometry

- Locally $H = dB$, i.e. $H_{ijk} = \partial_{\underline{i}} B_{\underline{jk}}$.
- Locally $R = [\beta, \beta]_{SN}$, i.e. $R^{ijk} = \beta^{\underline{i}n} \partial_n \beta^{\underline{jk}}$.

Can be implemented by *twists*, e.g. twisted Courant bracket.

2. DFT Only local expressions in the literature!

- $H_{ijk} = \partial_{\underline{i}} B_{\underline{jk}} + B_{\underline{i}n} \tilde{\partial}^n B_{\underline{jk}}$.
- $R^{ijk} = \tilde{\partial}^{\underline{i}} \beta^{\underline{jk}} + \beta^{\underline{i}n} \partial_n \beta^{\underline{jk}}$.

Two questions

- Is it possible to understand the objects $\langle , \rangle$, $\tilde{\partial}^k$ and $[,]_C$ using Poisson brackets on the cotangent bundle of an appropriate space?
- Is there a geometric interpretation of the DFT versions of H_{ijk} and R^{ijk}?

Lie algebroids and parity change

Liu, Mackenzie, Roytenberg, Severa, Vaintrob, Voronov, Weinstein, Xu,

Definition

A vector bundle $A \to M$ is called Lie algebroid, if there exists a homological vector field d_A on the supermanifold ΠA, i.e. $[d_A, d_A] = 0$.

Standard examples:

- $\underline{A = TM}$, basis of sections e_i, $[e_i, e_j]_A = f_{ij}^k e_k$ label coordinates on ΠA by (x^i, ξ^i), then

$$d_A = a_j^i(x)\xi^j\partial_i - \frac{1}{2}f_{ij}^k(x)\xi^i\xi^j\frac{\partial}{\partial\xi^k} \;. \tag{4}$$

- $\underline{A^* = T^*M}$, basis e^i, $[e^i, e^j]_{A^*} = Q_k^{ij}e^k$, label coordinates on ΠA^* by (x^i, θ_i), then

$$d_{A^*} = a^{ij}(x)\theta_i\partial_j - \frac{1}{2}Q_k^{ij}(x)\theta_i\theta_j\frac{\partial}{\partial\theta_k} \;. \tag{5}$$

The pair (A, A^*) is an example of a *Lie bialgebroid*.

Taking the cotangent bundle

Roytenberg, arXiv:math/9910078

Why? Idea: d_A gives ∂_i on functions. Relation of d_{A^*} to $\tilde{\partial}^i$? - If so, we need to get expressions like $\partial_i f + \tilde{\partial}^i f$, i.e. *define a meaningful sum of d_A and d_{A^*}!*

$T^*\Pi A$:
Coords: $(x^i, \xi^i, x^*_i, \xi^*_i)$ and Poisson structure:

$$\{x^j, x^*_i\} = \delta^j_i\ , \quad \{\xi^j, \xi^*_i\} = \{\xi^*_i, \xi^j\} = \delta^j_i\ .$$

- De Rham differential: Take Poisson bracket with the function

$$h_{d_A} = a^i_j(x) x^*_j \xi^i - \frac{1}{2} f^k_{ij} \xi^i \xi^j \xi^*_k\ , \tag{6}$$

- Poisson-Lichnerowicz-differential:

$$L^* h_{d_{A^*}} = a^{ij} \xi^*_i x^*_j - \frac{1}{2} Q_k{}^{ij} \xi^*_i \xi^*_j \xi^k\ , \tag{7}$$

Now add: $\mu := h_{d_A} + L^* h_{d_{A^*}}$.

The Drinfel'd double

Roytenberg, Weinstein

Theorem

A pair of Lie algebroids (A, A^*) is a Lie bialgebroid, iff $\{\mu, \mu\} = 0$.

Thus the following definition is justified:

Definition

The Drinfel'd double of a Lie bialgebroid (A, A^*) is given by $T^*\Pi A$ together with the homological vector field $\{\mu, \cdot\}$.

Application to double field theory

Deser, Stasheff

Two sets of momenta:

$$h_{d_A} = \xi^i\left(a_i^j x_j^* - \frac{1}{2} f_{ij}^k \xi^j \xi_k^*\right) =: \xi^i p_i \,, \qquad L^* \, h_{d_{A^*}} = \xi_i^*\left(a^{ij} x_j^* + Q_k^{ij} \xi^k \xi_j^*\right) =: \xi_i^* \tilde{p}^i \,. \tag{8}$$

Thus, we get two derivative operators for $f \in \mathcal{C}^\infty(M)$, seen as $f \in \mathcal{C}^\infty(T^*\Pi A)$:

$$\partial_i f := \{p_i, f\} \,, \quad \tilde{\partial}^i f := \{\tilde{p}^i, f\} \,, \tag{9}$$

More general: Lift of a generalized vector field:

$$V^m \partial_m + V_m dx^m \to V^m(x)\xi_m^* + V_m(x)\xi^m \ \in \mathcal{C}^\infty(T^*\Pi A) \,.$$

Now, what is the C-bracket?

C-bracket as derived bracket

Deser, Stasheff

Theorem

Let $V^m e_m + V_m e^m$ and $W^m e_m + W_m e^m$ be generalized vectors with corresponding lifts to $T^*\Pi A$ given by $V = V^m \xi^*_m + V_m \xi^m$ and $W = W^m \xi^*_m + W_m \xi^m$. In addition let the operation $\circ$ be defined by:

$$V \circ W = \left\{ \{\xi^i p_i + \xi^*_i \tilde{p}^i, V\}, W \right\},$$

Then, for vanishing fluxes f, Q, the C-bracket of V and W is given by

$$[V, W]_C = \frac{1}{2} \Big(V \circ W - W \circ V \Big) . \tag{10}$$

Thus, the C-bracket can be seen as a Courant bracket, written in a form appropriate to DFT.

Fibre translations on the Drinfel'd double

Roytenberg, 2002

Lifts of B and β to $T^*\Pi A$ define vector fields

$$X_B = \{B, \cdot\} = \{\tfrac{1}{2}B_{ij}\xi^i\xi^j, \cdot\}$$
$$X_\beta = \{\beta, \cdot\} = \{\tfrac{1}{2}\beta^{ij}\xi^*_i\xi^*_j, \cdot\}\,. \tag{11}$$

Interpretation: For $V = X + \alpha = X^i\xi^*_i + \alpha_i\xi^i$ we get

$$V \mapsto V + X_B V = \{\tfrac{1}{2}B_{ij}\xi^i\xi^j, X^i\xi^*_i + \alpha_i\xi^i\}$$
$$= V + V^m B_{mj}\xi^j\ ,$$

i.e. we get $V \mapsto V + \iota_X B$, i.e. B-transform. Similar for β.

Remark: Together with X_A for $A = A^i_j\xi^*_i\xi^j$, they generate the *Atiyah algebra* of the Drinfel'd double.

Fluxes in DFT

Idea: Twist the homological vector fields:

$$h_{d_A} \mapsto h_{d_A} + \{B, L^* h_{d_{A^*}}\} =: \ h_{d_A}^B \ . \qquad (12)$$

Then the H-flux is

$$\begin{aligned} H = \{h_{d_A}^B, B\} &= \ \{h_{d_A}, B\} + \{\{B, L^* h_{d_{A^*}}\}, B\} \\ &= \Big(\partial_i B_{jk} + B_{in}\tilde{\partial}^n B_{jk}\Big)\xi^i \xi^j \xi^k \ . \end{aligned}$$

i.e. the DFT expression.

Fluxes in DFT

Idea: Twist the homological vector fields:

$$L^* h_{d_{A^*}} \mapsto L^* h_{d_{A^*}} + \{\beta, h_{d_A}\} =: L^* h^{\beta}_{d_{A^*}} . \tag{13}$$

Then the R-flux is

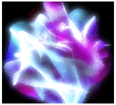
$$\begin{aligned} R &= \{L^* h^{\beta}_{d_{A^*}}, \beta\} = \{L^* h_{d_A}, \beta\} + \{\{\beta, h_{d_A}\}, \beta\} \\ &= \left(\beta^{in}\partial_n \beta^{jk} + \tilde{\partial}^i \beta^{jk}\right) \xi^*_i \xi^*_j \xi^*_k . \end{aligned}$$

i.e. the DFT expression.

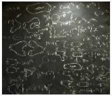
Outlook

- Interpretation of the C-bracket in terms of Poisson brackets on the Drinfel'd double
- Geometric understanding of the local structure of DFT fluxes
- **Q**: Relation to the phase space approach of Aschieri, Mylonas, Schupp, Szabo?
- **Q**: Deformation quantization leading to non-associative star products...is associativity restored if we apply the strong constraint?
- **Q**: Fluxed C-bracket?
- Many more...would be glad to discuss!

Next time...





Nordic String Theory Meeting 2017

Leibniz
Universität
Hannover

This workshop continues the tradition of the annual Nordic String Theory Meetings. As in the past, the idea is to have a short but intense meeting of stringy people from the "nordic" places Berlin(HU), Bremen(Jacobs), Copenhagen(NBI), Göttingen, Groningen, Hamburg(Uni/DESY), Hannover, Potsdam(AEI) and possibly some nordophilic people from not so nordic places.

Date:	February/March 2017 (to be determined)
Contact:	Olaf Lechtenfeld